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B Proofs (detailed version)

B.1 Proof of Proposition 1

Given any set of parameters ρ, τ, and ϕ, and other firms' choices, a firm i chooses y_i and Q_i to maximize profits, which reads as follows when plugging in the functional forms of demand and algorithm:

$$\pi_i(y_i, Q_i) = \frac{Q_i^\rho}{\sum_{k\in N} Q_k^\rho} \cdot y_i \cdot [(B(Q_i) - y_i)(1-\tau) - C(Q_i) - \phi]. \tag{B.1}$$

The feasible choices satisfy $Q_i \geq 0$ and $0 \leq y_i \leq B(Q_i)$, so that $p_i = B(Q_i) - y_i \geq 0$. We focus on $0 \leq \tau < 1$.[11]

We define the following value function:[12]

$$V(Q) := B(Q)(1-\tau) - C(Q) - \phi. \tag{B.2}$$

By strict concavity of $B(Q)$ and weak convexity of $C(Q)$, $V(Q)$ is a strictly concave function. Its unique maximum $\tilde{Q} := \arg\max_{Q\geq 0} V(Q)$ is attained either at $\tilde{Q} = 0$ or interior at some point $\tilde{Q} > 0$. Indeed, $V'(Q) < 0$ for sufficiently large Q by the maintained assumptions. Now, we distinguish two cases: (i) $V(\tilde{Q}) \leq 0$ and (ii) $V(\tilde{Q}) > 0$.

1. Case (i): Suppose $V(\tilde{Q}) \leq 0$. For $y_i > 0$, per-unit profit satisfies

   $$v(y_i, Q_i) = V(Q_i) - (1-\tau)y_i < 0.$$

   Since any feasible positive quantity requires $Q_i > 0$ and hence positive awareness, such a choice yields negative profits. Thus any best response has $y_i = 0$ and yields zero profits, and choosing $y_i = Q_i = 0$ for each firm $i \in N$ constitutes a symmetric Nash equilibrium.[13]

[11] If $\tau \geq 1$, choosing $y_i = Q_i = 0$ for every firm gives a symmetric equilibrium. Indeed, for $\tau \geq 1$, any feasible $y_i > 0$ requires $Q_i > 0$, so $C(Q_i) > 0$ and awareness is positive. Since $p_i \geq 0$, per-unit profit $p_i(1-\tau) - C(Q_i) - \phi$ is strictly negative. Hence every best response has $y_i = 0$, and there is no interior equilibrium in this case.

[12] The value function $V(Q)$ is strongly related to the per-unit profit function $v(y, Q)$, as $V(Q) = v(y, Q) + (1-\tau)y$. In particular, for $\tau < 1$ and $y \geq 0$, we have $V(Q) \geq v(y, Q)$, and $V'(Q) = \frac{\partial v(y,Q)}{\partial Q}$.

[13] In this case, a feasible strategy profile is a Nash equilibrium if and only if all quantities are zero, i.e., $y_i = 0$ for all $i \in N$; the qualities can be arbitrary.

2. Case (ii): $V(\tilde{Q}) > 0$. Since $V(0) = -\phi \leq 0$, we have $\tilde{Q} > 0$. Consider $y_i = \epsilon > 0$ and $Q_i = \tilde{Q}$. For ϵ sufficiently small, this choice is feasible and has strictly positive per-unit profit, since

$$v(\epsilon, \tilde{Q}) = V(\tilde{Q}) - (1-\tau)\epsilon > 0.$$

Moreover, profits are strictly positive, $\pi_i(\epsilon, \tilde{Q}) > 0$, as all three factors in (B.1) are positive. Therefore, any best response features positive profits, which requires $y_i > 0$ and $Q_i > 0$ (as $B(0) = 0$). The upper quantity boundary $y_i = B(Q_i)$ cannot yield positive profits either, since it implies $p_i = 0$.

The case distinction delivers a boundary solution for every firm in Case (i), and requires an interior solution for every firm in Case (ii). We continue with Case (ii), establish global best responses, exclude asymmetric equilibria, and finally show existence and uniqueness of a symmetric equilibrium.

Any interior best response must satisfy the two first order conditions Equations (2) and (3), which simplify to the equations below when using the functional form for the algorithm and the quasi-linear demand.

$$y_i = \frac{B(Q_i)(1-\tau) - C(Q_i) - \phi}{2(1-\tau)} \tag{B.3}$$

$$\frac{\rho \sum_{j \neq i} Q_j^\rho}{Q_i(Q_i^\rho + \sum_{j \neq i} Q_j^\rho)} = \frac{C'(Q_i) - B'(Q_i)(1-\tau)}{(B(Q_i) - y_i)(1-\tau) - C(Q_i) - \phi}. \tag{B.4}$$

Now, plugging (B.3) into (B.4) yields:

$$\frac{\rho \sum_{j \neq i} Q_j^\rho}{2Q_i(Q_i^\rho + \sum_{j \neq i} Q_j^\rho)} = \frac{C'(Q_i) - B'(Q_i)(1-\tau)}{B(Q_i)(1-\tau) - C(Q_i) - \phi}. \tag{B.5}$$

Denote the RHS by $R(Q) := -V'(Q)/V(Q)$. On the interval where $V(Q) > 0$,

$$R'(Q) = \frac{[V'(Q)]^2 - V''(Q)V(Q)}{V(Q)^2} > 0.$$

Strict concavity, $V(0) \leq 0$, and the eventually negative derivative imply that V has two roots $0 \leq \underline{Q} < \tilde{Q} < \bar{Q}$, with $V(Q) > 0$ exactly on $(\underline{Q}, \bar{Q})$.

To verify that (B.5) characterizes a global best response, fix the other firms' qualities. For $Q_i \in (\underline{Q}, \bar{Q})$, profits are strictly concave in y_i, and (B.3) gives their unique maximizer, which satisfies $0 < y_i < B(Q_i)$. The resulting reduced profit is

$$g_i(Q_i) := A(Q_i, Q_{-i}) \frac{V(Q_i)^2}{4(1-\tau)}.$$

If $\rho = 0$, awareness is constant, so the unique best response quality is $\tilde{Q}$. The same holds if $\rho > 0$ and all rivals choose zero quality, since then $A(Q_i, Q_{-i}) = 1$ for every $Q_i > 0$. Otherwise, let $S_i := \sum_{j \neq i} Q_j^\rho > 0$. Then

$$\frac{g_i'(Q_i)}{g_i(Q_i)} = 2\left[\frac{\rho S_i}{2Q_i(Q_i^\rho + S_i)} - R(Q_i)\right].$$

The first term in brackets is positive and strictly decreasing in Q_i, whereas R is strictly increasing, with $R(\tilde{Q}) = 0$ and $R(Q) \to \infty$ as $Q \uparrow \bar{Q}$. For $Q < \tilde{Q}$, $R(Q) < 0$. Hence the bracketed expression crosses zero exactly once, from above, at a quality in $(\tilde{Q}, \bar{Q})$. The reduced profit therefore increases up to this quality and decreases thereafter. Choices outside $(\underline{Q}, \bar{Q})$ cannot yield positive profits. This proves that (B.5), together with (B.3), characterizes the unique global best response in Case (ii), for every finite $\rho \geq 0$.

If $\rho = 0$, all firms consequently choose the same quality $\tilde{Q}$ and the same quantity given by (B.3), so the equilibrium is unique and symmetric. If $n = 1$, awareness is identically one for any ρ, and the same conclusion holds; equation (4) then also reduces to $R(Q) = 0$. For the remainder, suppose $\rho > 0$ and $n \geq 2$.

In any equilibrium in Case (ii), every firm has positive quality. Thus $S_i > 0$ and $R(Q_i) > 0$, and (B.5) implies $\rho > 2R(Q_i)Q_i$. It can be rewritten as

$$\begin{aligned}
\frac{\rho \sum_{j \neq i} Q_j^\rho}{2Q_i(Q_i^\rho + \sum_{j \neq i} Q_j^\rho)} &= R(Q_i) \\
\frac{2R(Q_i)Q_i^{1+\rho}}{\rho - 2R(Q_i)Q_i} &= \sum_{j \neq i} Q_j^\rho \\
\frac{Q_i^\rho}{\frac{\rho}{2R(Q_i)Q_i} - 1} &= \sum_{j \neq i} Q_j^\rho.
\end{aligned}$$

Suppose that there are two firms i, j such that $Q_i \neq Q_j$. We have that

$$\begin{aligned}
\frac{Q_i^\rho}{\frac{\rho}{2R(Q_i)Q_i} - 1} &= Q_j^\rho + \sum_{k \neq i,j} Q_k^\rho \\
\frac{Q_j^\rho}{\frac{\rho}{2R(Q_j)Q_j} - 1} &= Q_i^\rho + \sum_{k \neq i,j} Q_k^\rho.
\end{aligned}$$

Subtracting the two equations and rearranging yields

$$\frac{Q_i^\rho}{\rho - 2R(Q_i)Q_i} = \frac{Q_j^\rho}{\rho - 2R(Q_j)Q_j}. \tag{B.6}$$

Denote $\Phi(Q) := \frac{Q^\rho}{\rho - 2R(Q)Q}$. We have that

$$\frac{\partial \Phi}{\partial Q} = \frac{\rho Q^{\rho-1}[\rho - 2R(Q)Q] + Q^\rho[2R'(Q)Q + 2R(Q)]}{[\rho - 2R(Q)Q]^2}, \tag{B.7}$$

which is positive wherever $Q \geq \tilde{Q}$ and $\rho > 2R(Q)Q$. This domain is an interval, since $QR(Q)$ is strictly increasing on $[\tilde{Q}, \bar{Q})$, and it contains both candidate equilibrium qualities by (B.5). Hence (B.6) can hold only if $Q_i = Q_j$, a contradiction. Equation (B.3) then also implies equal quantities. This excludes any interior equilibrium that is asymmetric.

Now, consider symmetry $y_i = y > 0$ and $Q_i = Q > 0$ for all firms i. Equation (B.5) simplifies as follows (which equals Equation (4)):

$$\rho \frac{n-1}{2n} = Q \cdot \frac{C'(Q) - B'(Q)(1-\tau)}{B(Q)(1-\tau) - C(Q) - \phi}. \tag{B.8}$$

The RHS is $QR(Q)$, where $R(Q) = -V'(Q)/V(Q)$ and $V(Q) > 0$ on $(\underline{Q}, \bar{Q})$. It is negative below $\tilde{Q}$, zero at $\tilde{Q}$, and strictly increasing on $[\tilde{Q}, \bar{Q})$ because $R(Q) \geq 0$ and $R'(Q) > 0$ there. As $Q \uparrow \bar{Q}$, it tends to infinity. Therefore, for $\rho > 0$ and $n \geq 2$, there is a unique solution $Q^* \in (\tilde{Q}, \bar{Q})$ to (B.8).

Taken together, for any parameters ρ, τ, ϕ that induce Case (ii), there is a unique quality $Q^* > 0$ satisfying (B.8). Equation (B.3) delivers a unique corresponding quantity $y^* > 0$. By the global best-response argument above, every firm choosing $y_i = y^*$ and $Q_i = Q^*$ constitutes a Nash equilibrium. Since all equilibria in Case (ii) are interior and symmetric, this equilibrium is unique.

B.2 Proof of Theorem 1

By Proposition 1, any platform parameters inducing positive trade yield a unique symmetric equilibrium. In such an equilibrium, $Q > 0$, $0 \leq \tau < 1$, and

$$y = \frac{B(Q)(1-\tau) - C(Q) - \phi}{2(1-\tau)} > 0.$$

Moreover, equation (4) implies

$$\rho = \frac{2nQ}{n-1} \frac{C'(Q) - B'(Q)(1-\tau)}{B(Q)(1-\tau) - C(Q) - \phi}. \tag{B.9}$$

Thus, a triple (Q, τ, ϕ) with $Q > 0$, $0 \leq \tau < 1$, and $\phi \geq 0$ is implementable with positive trade if and only if

$$B(Q)(1-\tau) - C(Q) - \phi > 0, \qquad C'(Q) - B'(Q)(1-\tau) \geq 0.$$

Indeed, necessity follows from the equilibrium conditions, while sufficiency follows by choosing ρ according to (B.9) and applying Proposition 1.

We first rule out positive transaction fees, establish existence, and then characterize the optimum.

(i) Consider any positive-trade design with $\phi > 0$, inducing quality Q and quantity y. The quantity condition gives

$$(1-\tau)[B(Q) - 2y] = C(Q) + \phi.$$

Define an alternative design by $\phi' = 0$ and

$$1 - \tau' = \frac{C(Q)}{B(Q) - 2y} = \frac{(1-\tau)C(Q)}{C(Q) + \phi}.$$

Then $\tau < \tau' < 1$, and the alternative preserves the same quantity at quality Q. It also satisfies

$$C'(Q) - B'(Q)(1-\tau') > C'(Q) - B'(Q)(1-\tau) \geq 0.$$

Consequently, the same Q and y can be implemented by choosing a finite $\rho' > 0$ according to (B.9).

Using the quantity condition, platform profit can be written as

$$\Pi_{\text{Pl}} = [\tau(B(Q) - y) + \phi]y = [B(Q) - C(Q)]y - (2-\tau)y^2.$$

The alternative therefore increases platform profit by $(\tau' - \tau)y^2 > 0$. Hence every positive-trade design with a positive transaction fee is strictly dominated by one with $\phi = 0$.

(ii) With $\phi = 0$, the platform's reduced objective is

$$F(Q, \tau) = \frac{\tau}{4}\left[B(Q)^2 - \frac{C(Q)^2}{(1-\tau)^2}\right]. \tag{B.10}$$

For convenience, define

$$z(Q) := \frac{C(Q)}{B(Q)}, \qquad r(Q) := \frac{C'(Q)}{B'(Q)}.$$

The positive-trade feasible set is characterized by

$$Q > 0, \qquad z(Q) < 1 - \tau \leq 1, \qquad 1 - \tau \leq r(Q).$$

The maintained assumptions imply, for $Q > 0$,

$$C(Q) \leq QC'(Q), \qquad B(Q) > QB'(Q),$$

and hence $z(Q) < r(Q)$.

Let $Q_{\max} > 0$ be the unique positive solution to $B(Q_{\max}) = C(Q_{\max})$. It exists because $B - C$ is strictly concave, initially increasing, and eventually has a strictly negative derivative. Positive trade requires $Q < Q_{\max}$. For any $Q \in (0, Q_{\max})$, choosing

$$z(Q) < 1 - \tau < \min\{r(Q), 1\}$$

gives a feasible design with strictly positive platform profit. Thus no-trade designs and designs with both fees zero cannot be optimal.

To establish existence, observe that on the feasible set,

$$0 \leq F(Q, \tau) \leq \frac{B(Q)^2}{4}, \qquad 0 \leq F(Q, \tau) \leq \frac{1 - z(Q)}{4} B(Q)^2.$$

The first bound tends to zero as $Q \downarrow 0$, and the second tends to zero as $Q \uparrow Q_{\max}$. Fixing any feasible positive profit level, we can therefore restrict attention to qualities in a closed interval strictly inside $(0, Q_{\max})$. On this interval, close the feasible set by allowing $1 - \tau = z(Q)$, where $F = 0$. The resulting feasible set is compact and F is continuous. Thus a strictly positive maximum is attained. By part (i), this is also a global maximum of the original platform problem. Every optimum satisfies

$$\phi^* = 0, \qquad 0 < \tau^* < 1, \qquad z(Q^*) < 1 - \tau^*.$$

We next show that $\rho^* > 0$. Suppose instead that an optimum has $\rho = 0$, so that $C'(Q) = B'(Q)(1 - \tau)$. Differentiating (B.10) with respect to Q, holding

τ fixed, gives at this boundary

$$F_Q(Q,\tau) = \frac{\tau B'(Q)}{2}\left[B(Q) - \frac{C(Q)}{1-\tau}\right] > 0.$$

Furthermore,

$$r'(Q) = \frac{C''(Q)B'(Q) - C'(Q)B''(Q)}{B'(Q)^2} > 0.$$

A sufficiently small increase in Q, holding τ fixed, therefore preserves both positive trade and implementability, and raises profit. This contradicts optimality. Hence $\rho^* > 0$.

(iii) At any optimum, all constraints of the reduced problem are locally slack. We may therefore differentiate with respect to Q and τ, treating them as independent choice variables. Taking logs of (B.10), and omitting the constant $-\ln 4$, gives

$$\ln F = -2\ln(1-\tau) + \ln\tau + \ln[B(Q)(1-\tau) + C(Q)] + \ln[B(Q)(1-\tau) - C(Q)].$$

The first-order conditions are

$$\begin{aligned} \frac{B'(Q)(1-\tau) + C'(Q)}{B(Q)(1-\tau) + C(Q)} - \frac{C'(Q) - B'(Q)(1-\tau)}{B(Q)(1-\tau) - C(Q)} &= 0 \qquad [Q], \\ \frac{2}{1-\tau} + \frac{1}{\tau} - \frac{B(Q)}{B(Q)(1-\tau) + C(Q)} - \frac{B(Q)}{B(Q)(1-\tau) - C(Q)} &= 0 \qquad [\tau]. \end{aligned}$$

The condition $[\tau]$ yields

$$\begin{aligned} [B(Q)]^2(1-\tau)^3 &= [C(Q)]^2(1+\tau), \\ \left[\frac{B(Q)}{C(Q)}\right]^2 &= \frac{1+\tau}{(1-\tau)^3}. \end{aligned}$$

From $[Q]$, we also get

$$\begin{aligned} \frac{B'(Q)(1-\tau) + C'(Q)}{B(Q)(1-\tau) + C(Q)} &= \frac{C'(Q) - B'(Q)(1-\tau)}{B(Q)(1-\tau) - C(Q)}, \\ B'(Q)B(Q)(1-\tau)^2 &= C'(Q)C(Q), \\ \frac{B(Q)}{C(Q)} &= \frac{C'(Q)}{B'(Q)}\frac{1}{(1-\tau)^2}, \\ \left[\frac{B(Q)}{C(Q)}\right]^2 &= \left[\frac{C'(Q)}{B'(Q)}\right]^2 \frac{1}{(1-\tau)^4}. \end{aligned}$$

Combining the two expressions yields

$$\tau^* = \sqrt{1 - \left[\frac{C'(Q^*)}{B'(Q^*)}\right]^2}. \tag{B.11}$$

Since $\tau^* > 0$, this implies $C'(Q^*) < B'(Q^*)$ and hence $Q^* < \hat{Q}$.

Finally, substituting (B.11) and $\phi = 0$ into (4) gives

$$\rho^* = \frac{2n}{n-1}\, Q^* \frac{C'(Q^*) - B'(Q^*)(1-\tau^*)}{B(Q^*)(1-\tau^*) - C(Q^*)}. \tag{B.12}$$

(iv) It remains to establish uniqueness. Define

$$E(Q) := \frac{C'(Q)B(Q)}{B'(Q)C(Q)} = \frac{r(Q)}{z(Q)}.$$

The two first-order conditions in part (iii) imply

$$r(Q)z(Q) = (1-\tau)^2, \qquad z(Q)^2 = \frac{(1-\tau)^3}{1+\tau}.$$

Consequently, at every optimum,

$$E(Q) = \frac{1+\tau}{1-\tau}, \qquad 1 - \tau = \frac{2}{1+E(Q)}.$$

Substituting the latter expression into the equation for $z(Q)^2$ yields

$$z(Q)\sqrt{E(Q)}[1 + E(Q)] = 2.$$

Moreover,

$$z'(Q) = \frac{B'(Q)}{B(Q)}[r(Q) - z(Q)] > 0.$$

By the additional assumption, $E(Q)$ is non-decreasing on $(0, \hat{Q})$. Therefore the left-hand side of the preceding equation is strictly increasing on this interval, which contains every optimal quality. Hence there is at most one optimal quality Q^*. Existence was established in part (ii), and (B.11) and (B.12) then uniquely determine τ^* and ρ^*, while $\phi^* = 0$.

□

Proof of Proposition 2

By Theorem 1, a platform-optimal design satisfies $\rho^* > 0$, $0 < \tau^* < 1$, and $\phi^* = 0$. Throughout the proof, set $\phi = 0$, and let Q and y denote the symmetric equilibrium choices characterized in Proposition 1.

For convenience, write $a := 1 - \tau$, denote the specialization of the function in (A.2) to $\phi = 0$ by V_0, and define

$$V_0(Q) := aB(Q) - C(Q), \qquad G(Q) := C'(Q) - aB'(Q), \qquad H(Q, \tau) := \frac{QG(Q)}{V_0(Q)}.$$

Derivatives of V_0 and G with respect to Q hold τ fixed. In a neighborhood of the platform optimum, positive trade and $\rho > 0$ imply $V_0(Q) > 0$ and $G(Q) > 0$. Moreover,

$$V_0'(Q) = -G(Q), \qquad G'(Q) = C''(Q) - aB''(Q) > 0.$$

Equations (4) and (A.3) give

$$H(Q, \tau) = \rho \frac{n-1}{2n}, \qquad y = \frac{V_0(Q)}{2a}.$$

In particular,

$$H_Q = \frac{G(Q) + QG'(Q)}{V_0(Q)} + \frac{QG(Q)^2}{V_0(Q)^2} > 0,$$

so the equilibrium quality is locally differentiable in both parameters. In the symmetric equilibrium, consumer surplus and aggregate firm profits are

$$CS = \frac{y^2}{2}, \qquad \sum_{i \in N} \pi_i = ay^2.$$

Social welfare is

$$W = CS + \sum_{i \in N} \pi_i + \Pi^{\mathrm{Pl}}.$$

(i) Hold $\tau = \tau^*$ fixed. Differentiating the equilibrium condition with respect to ρ yields

$$\frac{dQ}{d\rho} = \frac{n-1}{2nH_Q} > 0.$$

Consequently,

$$\frac{dy}{d\rho} = -\frac{G(Q)}{2a} \frac{dQ}{d\rho} < 0.$$

Thus reducing ρ lowers quality and increases quantity. It also increases con-

sumer surplus and aggregate firm profits, since

$$\frac{dCS}{d\rho} = y\frac{dy}{d\rho} < 0, \qquad \frac{d}{d\rho}\sum_{i\in N}\pi_i = 2ay\frac{dy}{d\rho} < 0.$$

Since $\rho^* > 0$ is an interior platform-optimal choice,

$$\left.\frac{d\Pi^{\text{Pl}}}{d\rho}\right|_* = 0,$$

where the derivative incorporates the firms' equilibrium responses and holds the other platform parameters fixed. Therefore,

$$\left.\frac{dW}{d\rho}\right|_* = (3-2\tau^*)y^*\left.\frac{dy}{d\rho}\right|_* < 0.$$

(ii) Hold $\rho = \rho^*$ fixed. The partial derivative of H with respect to τ, holding Q fixed, is

$$H_\tau = \frac{Q[B'(Q)V_0(Q) + B(Q)G(Q)]}{V_0(Q)^2} > 0.$$

Implicit differentiation therefore gives

$$\frac{dQ}{d\tau} = -\frac{H_\tau}{H_Q} < 0.$$

Thus reducing the value fee increases equilibrium quality. Differentiating equilibrium quantity gives

$$\frac{dy}{d\tau} = -\frac{C(Q)}{2a^2} - \frac{G(Q)}{2a}\frac{dQ}{d\tau}.$$

Substituting the expression for $dQ/d\tau$ and simplifying yields

$$\frac{dy}{d\tau} = -\frac{C(Q)G(Q) + Q[C(Q)G'(Q) - G(Q)C'(Q)]}{2a^2V_0(Q)H_Q}.$$

The denominator is positive. The numerator satisfies

$$C(Q)G(Q) + Q[C(Q)G'(Q) - G(Q)C'(Q)] = C(Q)^2\frac{d}{dQ}\left(\frac{QG(Q)}{C(Q)}\right).$$

Furthermore,

$$\frac{QG(Q)}{C(Q)} = \varepsilon_C(Q) - a\frac{QB'(Q)}{C(Q)}.$$

By convexity of C and $C(0) = 0$, $\varepsilon_C(Q) \geq 1$. Together with $B''(Q) < 0$, this

implies

$$\frac{d}{dQ}\left(\frac{QB'(Q)}{C(Q)}\right) = \frac{B'(Q)}{C(Q)}\left[1 + \frac{QB''(Q)}{B'(Q)} - \varepsilon_C(Q)\right] < 0.$$

Hence the assumption $\varepsilon'_C(Q^*) \geq 0$ implies

$$\frac{d}{dQ}\left(\frac{QG(Q)}{C(Q)}\right)\bigg|_* > 0, \qquad \frac{dy}{d\tau}\bigg|_* < 0.$$

It follows that

$$\frac{dCS}{d\tau}\bigg|_* = y^* \left.\frac{dy}{d\tau}\right|_* < 0,$$

and

$$\frac{d}{d\tau}\sum_{i\in N}\pi_i\bigg|_* = -(y^*)^2 + 2(1-\tau^*)y^* \left.\frac{dy}{d\tau}\right|_* < 0.$$

Since $0 < \tau^* < 1$ is an interior platform-optimal choice,

$$\frac{d\Pi^{\mathrm{Pl}}}{d\tau}\bigg|_* = 0,$$

holding $\rho = \rho^*$ and $\phi = 0$ fixed and incorporating the firms' equilibrium responses. Consequently,

$$\frac{dW}{d\tau}\bigg|_* = (3-2\tau^*)y^* \left.\frac{dy}{d\tau}\right|_* - (y^*)^2 < 0.$$

The strict derivative signs establish the stated changes for sufficiently small reductions from the platform-optimal design. □

B.3 Proof of Proposition 3

If $\bar{\tau} \geq \tau^*$, the unconstrained optimum remains feasible and is the unique solution by Theorem 1. For the remainder, consider a binding restriction $\bar{\tau} < \tau^*$.

Write $a := 1 - \tau$ and define

$$S(Q) := B(Q) - C(Q), \qquad z(Q) := \frac{C(Q)}{B(Q)}, \qquad r(Q) := \frac{C'(Q)}{B'(Q)}, \qquad \phi(Q) := r(Q)z(Q).$$

By the maintained assumptions,

$$C(Q) \leq QC'(Q), \qquad B(Q) > QB'(Q),$$

so $z(Q) < r(Q)$ for $Q > 0$. Furthermore,

$$z'(Q) = \frac{B'(Q)}{B(Q)}[r(Q) - z(Q)] > 0, \qquad r'(Q) = \frac{C'''(Q)B'(Q) - C'(Q)B''(Q)}{B'(Q)^2} > 0.$$

Consequently, ϕ is strictly increasing.

We first solve the platform's problem for a fixed value fee, and then optimize the value fee subject to the cap.

(i) Fix $a = 1 - \tau > 0$. By Proposition 1, a positive-trade symmetric equilibrium satisfies

$$\phi = a[B(Q) - 2y] - C(Q),$$

and

$$\rho = \frac{2nQ}{n-1}\frac{C'(Q) - aB'(Q)}{2ay}.$$

Thus a pair (Q, y) is implementable with non-negative fees and quality rewards if and only if

$$Q > 0, \qquad 0 < y \leq \frac{aB(Q) - C(Q)}{2a}, \qquad r(Q) \geq a.$$

Necessity follows from the equilibrium conditions. Conversely, the displayed expressions for ϕ and ρ implement any such pair by Proposition 1.

Substituting for ϕ, platform profit becomes

$$\Pi_{\text{Pl}} = [(1-a)(B(Q) - y) + \phi]y = S(Q)y - (1+a)y^2.$$

For fixed Q and a, this expression is strictly concave in y, with unconstrained maximizer $S(Q)/[2(1+a)]$. Therefore,

$$y(Q, a) = \min\left\{\frac{S(Q)}{2(1+a)}, \frac{aB(Q) - C(Q)}{2a}\right\}.$$

The first quantity is feasible precisely when $z(Q) \leq a^2$. After optimizing quantity, platform profit is consequently

$$J(Q, a) = \begin{cases} \dfrac{S(Q)^2}{4(1+a)}, & z(Q) \leq a^2, \\ \dfrac{1-a}{4}\left[B(Q)^2 - \dfrac{C(Q)^2}{a^2}\right], & z(Q) \geq a^2, \end{cases}$$

on the positive-trade feasible domain. The two expressions coincide when

$z(Q) = a^2$. The first branch implements

$$\phi = \frac{a^2 B(Q) - C(Q)}{1 + a},$$

whereas the second implements $\phi = 0$.

(ii) Define

$$a^* := 1 - \tau^*, \qquad \widetilde{a} := \sqrt{z(\hat{Q})} = 1 - \widetilde{\tau}.$$

Theorem 1 gives

$$\phi(Q^*) = (a^*)^2, \qquad Q^* < \hat{Q}.$$

Since $r(Q^*) < 1$ and z is strictly increasing,

$$(a^*)^2 = r(Q^*)z(Q^*) < z(Q^*) < z(\hat{Q}).$$

Thus $a^* < \widetilde{a} < 1$, or equivalently $0 < \widetilde{\tau} < \tau^*$.

For $a \in [a^*, \widetilde{a})$, let $Q(a)$ be the unique solution of $\phi(Q(a)) = a^2$. This solution exists and lies in $[Q^*, \hat{Q})$ because ϕ is strictly increasing and $\phi(\hat{Q}) = z(\hat{Q}) = \widetilde{a}^2$.

On the first branch of J,

$$J_Q(Q, a) = \frac{S(Q)[B'(Q) - C'(Q)]}{2(1 + a)},$$

which is positive below $\hat{Q}$ and negative above it. On the second branch, for $a < 1$,

$$J_Q(Q, a) = \frac{1 - a}{2} B(Q)B'(Q) \left[1 - \frac{\phi(Q)}{a^2}\right].$$

At $Q(a) < \hat{Q}$, we have $r(Q(a)) < 1$ and hence

$$z(Q(a)) = \frac{a^2}{r(Q(a))} > a^2.$$

Thus $Q(a)$ belongs to the second branch. Any quality on the first branch lies below $Q(a)$, where that branch is increasing. On the second branch, profit increases up to $Q(a)$ and decreases thereafter. Hence $Q(a)$ is the unique global maximizing quality.

It is also implementable: since $r(Q(a))z(Q(a)) = a^2$ and $r(Q(a)) > z(Q(a))$,

$$z(Q(a)) < a < r(Q(a)).$$

Thus trade is positive, $\phi = 0$, and $\rho > 0$.

For $a \in [\widetilde{a}, 1]$, the quality $\hat{Q}$ belongs to the first branch, since $z(\hat{Q}) \leq a^2$. That branch is uniquely maximized at $\hat{Q}$. Any quality on the second branch satisfies $Q \geq \hat{Q}$, and

$$\phi(Q) = r(Q)z(Q) \geq a^2,$$

so, for $a < 1$, the second branch is strictly decreasing wherever it is present above $\hat{Q}$. For $a = 1$, the second branch has no positive-trade feasible points, since it requires $z(Q) \geq 1$, whereas positive trade requires $z(Q) < 1$. Hence the unique global maximizing quality is $\hat{Q}$. It is implementable because $r(\hat{Q}) = 1 \geq a$ and $z(\hat{Q}) = \widetilde{a}^2 < a$.

At $a = \widetilde{a}$, both branches yield $Q = \hat{Q}$ and $\phi = 0$, so their solutions coincide.

(iii) Let $\mathcal{J}(a)$ denote platform profit after optimizing quality and quantity. For $a \in [a^*, \widetilde{a})$, the envelope theorem and $\phi(Q(a)) = a^2$ give

$$\begin{aligned}\mathcal{J}'(a) &= \frac{1}{4}\left[-B(Q(a))^2 + \frac{C(Q(a))^2(2-a)}{a^3}\right] \\ &= \frac{B(Q(a))^2}{4}\left[\frac{2-a}{aE(Q(a))} - 1\right].\end{aligned}$$

At the unconstrained optimum, the first-order conditions from Theorem 1 imply

$$a^*[1 + E(Q^*)] = 2.$$

Since $Q(a)$ is increasing and E is non-decreasing on $(0, \hat{Q})$,

$$a[1 + E(Q(a))] > 2 \qquad \text{for } a \in (a^*, \widetilde{a}).$$

Thus $\mathcal{J}'(a) < 0$ on this interval.

For $a \in [\widetilde{a}, 1]$,

$$\mathcal{J}(a) = \frac{S(\hat{Q})^2}{4(1+a)},$$

which is strictly decreasing as well. Since the cap $\bar{\tau} < \tau^*$ requires $a \geq 1 - \bar{\tau} > a^*$, the platform uniquely chooses $a = 1 - \bar{\tau}$, or $\tau_c = \bar{\tau}$.

A zero-commission design with positive transaction fees generates positive profit and is feasible under every cap, so no-trade designs cannot be optimal. The preceding maximization therefore establishes the global solution to the capped problem.

The formulas in part (ii) of the proposition follow from $Q_c = Q(a)$ and $\phi_c = 0$.

In part (iii), $Q_c = \hat{Q}$ and

$$\phi_c = \frac{a^2 B(\hat{Q}) - C(\hat{Q})}{1+a}, \qquad y_c = \frac{S(\hat{Q})}{2(1+a)}.$$

Using $C'(\hat{Q}) = B'(\hat{Q})$ in the implementation formula for ρ gives

$$\rho_c = \frac{2n}{n-1} \frac{\hat{Q} B'(\hat{Q})(1-a^2)}{a S(\hat{Q})}.$$

These formulas also establish continuity at $a = \widetilde{a}$ and the stated zero-cap outcome.

(iv) Finally, consider social welfare along the capped solution:

$$W = S(Q_c) y_c - \frac{y_c^2}{2}.$$

In the moderately binding regime, $\phi(Q_c) = a^2$ with $a = 1 - \bar{\tau}$, so

$$\frac{dQ_c}{d\bar{\tau}} = -\frac{2a}{\phi'(Q_c)} < 0.$$

Since $y_c = [aB(Q_c) - C(Q_c)]/(2a)$,

$$\frac{dy_c}{d\bar{\tau}} = -\frac{C(Q_c)}{2a^2} + \frac{C'(Q_c) - aB'(Q_c)}{\phi'(Q_c)}.$$

For the following calculations, suppress the argument Q_c. The assumption that $E = r/z$ is non-decreasing implies $r'z \geq rz'$. Hence

$$\phi' = r'z + rz' \geq 2rz' = 2r\frac{B'}{B}(r - z).$$

Using $rz = a^2$ and $z < a < r$, we obtain

$$\frac{dy_c}{d\bar{\tau}} \leq -\frac{B}{2r} + \frac{B(r-a)}{2r(r-z)} = \frac{B}{2r}\frac{z-a}{r-z} < 0.$$

Thus tightening the cap increases both quality and quantity. Since $Q_c < \hat{Q}$ in the interior of this regime,

$$S'(Q_c) > 0, \qquad S(Q_c) - y_c > 0.$$

It follows that

$$\frac{dW}{d\bar{\tau}} = S'(Q_c)y_c\frac{dQ_c}{d\bar{\tau}} + [S(Q_c) - y_c]\frac{dy_c}{d\bar{\tau}} < 0.$$

In the strict-cap regime, $Q_c = \hat{Q}$ and

$$y_c = \frac{S(\hat{Q})}{2(2-\bar{\tau})}, \qquad \frac{dy_c}{d\bar{\tau}} = \frac{S(\hat{Q})}{2(2-\bar{\tau})^2} > 0.$$

Therefore,

$$\frac{dW}{d\bar{\tau}} = [S(\hat{Q}) - y_c]\frac{dy_c}{d\bar{\tau}} > 0.$$

For nonbinding caps the allocation, and hence welfare, is unchanged. Continuity at the regime boundaries and the strict inequalities establish that welfare is strictly increasing on $[0, \widetilde{\tau}]$, strictly decreasing on $[\widetilde{\tau}, \tau^*]$, and constant for $\bar{\tau} \geq \tau^*$. Consequently, welfare is uniquely maximized at $\bar{\tau} = \widetilde{\tau}$.

□